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A Study on Solutions of Perturbed Equations

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Abstract— Perturbation means a small disturbance in a physical system. Mathematically "Perturbation method" is a method for obtaining solution to complex equations (algebraic or differential) for which exact solution is not easy to find. In this paper, we discuss the applicability of asymptotic expansion method for singularly and regularly algebraic and ordinary differential equations. At first, we apply the method of asymptotic expansion for regular and singular pertubation equations (algebraic and differential) containing ε (0<ε<<1) and we conclude that the regular perturbed equtions give good approximation through this method. But, in case of singular perturbation equations this method is not successful so we introduce other methods like Matched Asymptotic Expansion Method, WKB Method. In this paper we are intrested in asymptotical analysis since this method tries to gain insight into the qualitative behaviour of a family of problems.

Index Terms— Matched Asymptotic Expansion Method, Regular Perturbation and Singular Perturbation, WKB Method

1 Introduction

Consider a diferential equation

$$f\left(x, y, \frac{dy}{dx}, \varepsilon\right) = 0 \quad (1.1)$$

with initial or boundary conditions, where x is independent variable, y is dependent variable and ε<<1 is the small parameter.

Define an asymptotic series,

$$y = y_0 + \varepsilon\, y_1 + \varepsilon^2\, y_2 + \ldots\ldots \quad (1.2)$$

where $y_0, y_1, y_2, \ldots$ are sufficiently smooth functions. To get the value of $y_0, y_1, y_2, \ldots$ we have to substitute (1.2) in (1.1) after doing the term by term differentiation. After substitution, we may get a sequence of problems and solvingfor first few terms, we will get $y_0, y_1, y_2, \ldots$. Such solutions obtained form an asymptotic series is called an asymptotic solution.

Numerical analysis and asymptotic analysis are the main methods for solving perturbation problem. The numerical analysis tries to provide quantitative information about a particular problem, whereas the asymptotic analysis gives the qualititative behavior of the family of problems.

Asymptotic analysis is powerful tool for finding analytical approximate solutions to complicated practical problems, which is an important branch of applied mathematics. In 1886 the establishment of rigorous foundation was done by Poincare and Stieltjes.

The main purpose of this study is to describe the application of perturbation expansion techniques to the solution of differential equations. The basic idea in perturbation theory is to obtain an approximate solution of a mathematical problem by exploiting the presence of a small dimensionless parameter –the smaller the parameter, the more accurate the approximate solution.

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2. BASIC DEFINITIONS

2.1

The problem which does not contain any small parameter is known as unperturbed problem.

Example: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x^2 - 8x + 4$, $y(0) = 3$, $\frac{dy}{dx}(0) = 3$

2.2

The problem which contains a small parameter is known as perturbed problem.

Example: $\frac{dy}{dx} + y = \varepsilon y^2$, $y(0) = 1$.

Depending upon the nature of perturbation, a perturbed problem can be divided into two categories.

1. Regular Perturbation
2. Singular Perturbation

2.3

The perturbation problem is said to be regular in nature, when the order (degree) of the perturbed and the unperturbed problem are same, when we set ε = 0. Generally, the parameter presented at lower order terms

2.4

The perturbed problem is said to be singularly perturbed, when the order (degree) of the problem is reduced when we set ε = 0.Generlly, the parameter presented at higher order terms and the lower order terms starts to dominate. Sometime the above statement is considered as the definition of singular perturbation problem.

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2.5

The letters 'O' and 'o' are order symbols. They are used to describe the rate at which the function approaches to limit value.

If a function f(x) approaches to a limit value at the same rate of another function g(x) at $x\to x_0$,then we can write $f(x)=O(g(x))$ as $x\to x_0$. We can write it as $\lim_{x\to x_0}\frac{f(x)}{g(x)}=C$, where C is finite. We can say here **"f is big-oh of g"**. If the expression $f(x) = o(g(x))$ as $x\to x_0$ means $\lim_{x\to x_0}\frac{f(x)}{g(x)}=0$. We can say here **"f is little-oh of g"** $x\to x_0$ and f(x) is smaller than g(x) as $x\to x_0$.

"Big-oh" and **"Little-oh"** notation are generally called "Landau" symbols. The expression $f(x) \sim g(x)$ as $x\to x_0$ means $\lim_{x\to x_0}\frac{f(x)}{g(x)}=1$ is called "f is asymptotically equal or approximately equal to g".

3 MATHEMATICAL MODEL

Consider a particle of mass M which is projected vertically upward with an initial speed Y_0. Let Y denote the speed at some general time T. If the air resistance is neglected then by Newton's second law of motion

$$M\frac{dY}{dT} = -Mg$$

Integrating we get, $Y = Y_0 - gT$, where the initial condition is $Y(0) = Y_0$.

On defining the non dimensional velocity v, and the time t, by $v = Y/Y_0$ and $t = gt/Y_0$, the given equation becomes

$$\frac{dv}{dt} = -1,\ v(0) = 1.$$

With the solution $v(t) = 1 - t$.

Taking account of the air resistance and proceed as above we get,

$$\frac{dv}{dt} = -1 - \varepsilon v$$
$$v(0) = 1$$

where $\varepsilon > 0$ is "small" parameter as the disturbances are very small.

4 PERTURBATION TECHNIQUES

In this section we discuss some test problems of both regular and singular type. The method of asymptotic expansion is applied and approximate solutions are obtained. Then it is compared with the exact solution.

4.1 REGULARLY PERTURBED PROBLEM:

Test Problem 4.1.1.
Consider the algebraic equation $y^2-\varepsilon y-1=0$. (4.1)
Let the roots ,

$$\alpha=1+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{8}+\cdots,\ \beta=-1+\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}+\cdots$$

Now, we try to find an approximate roots. Take an asymptote series $= y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots$, substitute in (4.1) and comparing the coefficients, we get

$$\alpha=1+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{8}+\cdots,\ \beta=-1+\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}+\cdots$$

we observed that the approximate roots obtained by an asymptotic expansion coincide with exact roots. For a regularly perturbed problem. We can very well obtain a good approximation to the exact root by an asymptotic series.

Test Problem 4.1.2.
Consider a perturbed differential equation of first order

$$\frac{dy}{dx}+y=\varepsilon y^2.\ y(0)=1. \quad (4.2)$$

Solving exactly and applying initial condition, we have

$$y=e^{-x}+\varepsilon(e^{-x}-e^{-2x})+\varepsilon^2(e^{-x}-e^{-2x}+e^{-3x})+\cdots \quad (4.3)$$

which is the exact solution of (4.2). Let us take an asymptotic series depends on independent variable x and small parameter ε is given by

$$y=\ y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots \quad (4.4)$$

substituting (4.4) in (4.2) and comparing the coefficients we get,

$$y=e^{-x}+\varepsilon(e^{-x}-e^{-2x})+\varepsilon^2(e^{-x}-e^{-2x}+e^{-3x})+\cdots \quad (4.5)$$

We observe that the exact solution of (4.2) given by (4.3) and approximate solution given by (4.5) are very well matching. The problem (4.2) is a regular perturbed differential equation. Also, if we put ε=0 the (4.2) becomes $\frac{dy}{dx}+y=0$. Comparing (4.2) and (4.5). We observe that the order of the perturbed differential equation and unperturbed differential equations are same.

4.2 SINGULARLY PERTURBED PROBLEM

Test Problem 4.2.1.
Consider an algebraic equation

$$\varepsilon x^2-x+1=0 \quad (4.6)$$

The roots of (4.6) are

$$\alpha=\frac{1}{\varepsilon}-1-\varepsilon-\cdots,\ \beta=1+\varepsilon+\varepsilon^2+\cdots$$

which are the two roots of (4.6). Consider an asymptotic series $x=x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots$(4.7). Substitute (4.7) in (4.6) and comparing the coefficients we get

$$x=1+\varepsilon+2\varepsilon+2\varepsilon^2+\cdots, \quad (4.7)$$

which matches to the solved root β up to two terms of the asymptotic expansion. So, we can get only one approximate root through asymptotic expansion. The other root α can not be approximated. So, for singularly perturbed problem, we cannot get good approximation to the exact roots by one term asymptotic series.

Test Problem 4.2.2.
Consider a differential equation

$$\varepsilon\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0,\ y(0)=0,\ y(1)=1. \quad (4.8)$$

Solving exactly and applyiong the boundary conditions we get, $y=\frac{e^{m_1x}-e^{m_2x}}{e^{m_1}-e^{m_2}}$

Which is the solution of (4.8), where $m_{1,2}=\frac{-1\pm\sqrt{(1+4\varepsilon)}}{2}$

Let us take an asymptotic series

$$y=\ y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots \quad (4.9)$$

Substituting (4.9) in (4.8) we get,

$$y_0(x)=\ Ce^{x} \quad (4.10)$$

the solution fails to satisfy one of the boundary condition. Again, we take ε = 0 in differential equation then, we get a first order differential equation

$$\frac{dy}{dx} = y = 0 \quad (4.11)$$

Where the order of the perturbed differential equation (4.8) and the unperturbed differential equation (4.11) are different. We conclude here by comparing both the solution that, we cannot get a good approximation to the exact solution by one term asymptotic series.

It is observed that the asymptotic expansion technique gives approximate solution which matches well to the exact solution for problem of regularity perturbed type. But at the same time, perturbation expansion does not provide a good approximation for singularly perturbed problems. One can use the method of matched asymptotic expansion and WKB method.

4.3 MATCHED ASYMPTOTIC EXPANSION METHOD

The method of matched asymptotic expansion was introduced by Ludwig Prandtl's boundary layer theory, in 1905. Mathematically, boundary layer occurs when small parameter is multiplying with highest derivative of diferential equation which is of singularly perturbed in nature. Depending upon the presence of boundary layer, the domain of the problem can be divided into two regions, first one is the outer region away from the boundary layer where the solution behaves smoothly and second one is the inner region where the gradient of the solution changes rapidly.

The algorithm for matched asymptotic expansion is given below.

Step-1: We will construct one solution in the outer region through asymptotic expansion away from the boundary layer.

Step-2: On the other hand, we will obtain another solution using stretching variable through asymptotic expansion within the boundary layer.

Step-3: We will match the leading order term of both the solutions by using them using Prandtl's matching condition which is given by

$$\lim_{x \to 0} f^{out}(x) = \lim_{x \to \infty} f^{in}(s).$$

4.4 WKB METHOD

When we can use the WKB Method?

Before we describe the exact situations where we can use the WKB Method, let's first consider an example.

$$\varepsilon^2 \frac{d^2y}{dx^2} = y = 0, \; 0 < \varepsilon << 1 \; (\varepsilon \text{ is small})$$

Note that there are two solutions to this differential equations

$$y(x,\varepsilon) = \exp(\pm \frac{ix}{\varepsilon})$$

since,

$$\frac{dy}{dx} = \pm \frac{i}{\varepsilon} \exp(\pm \frac{ix}{\varepsilon})$$

and

$$\frac{d^2y}{dx^2} = \frac{i^2}{\varepsilon^2} \exp(\pm \frac{ix}{\varepsilon}) = \frac{-1}{\varepsilon^2} \exp(\pm \frac{ix}{\varepsilon})$$

These are the type ofproblems that are ideal for the WKB Method because we want to be able to assume that the solution is in a specific form.

In general, we can use the WKB Method to solve problems in the following form.

$$\varepsilon^2 \frac{d^2y}{dx^2} + F(x)\, y = 0, \; 0 < \varepsilon << 1 \; ,$$

where F(x) is smooth and positive.

Presentation of WKB Method.

We will now show how to solve the following general differential equation.

$$\varepsilon^2 \frac{d^2y}{dx^2} + F(x)y = 0, 0 < \varepsilon << 1 \quad 4.4.1)$$

Using WKB Method.

As discussed in the above section, we will assume that the solution to this differential equation is of the form,

$$y(x,\varepsilon) = A(x,\varepsilon) \exp(\frac{iu(x)}{\varepsilon})$$

Then

$$\frac{dy}{dx} = (A' + A\frac{IU}{\varepsilon}) \exp(\frac{iu}{\varepsilon})$$

And

$$\frac{d^2y}{dx^2} = (-\frac{(-u')^2}{\varepsilon^2} A + \frac{i}{\varepsilon}(Au'' + 2A'u') + A'' + \exp(\frac{iu}{\varepsilon})$$

Substituting the above equations in (4.4.1) and comparing the coefficients we get

$$y(x,\varepsilon) \sim F(x)^{-1/4} \exp(\pm \frac{i}{\varepsilon} \int_{x_0}^{x} \sqrt{(F(s))} ds)$$

CONCLUSION

The asymptotic expansion method gives very good result for regularly perturbed problems where as one term asymptotic expansion does not work well for singularly perturbed problems. We conclude that, by the method of matched asymptotic expansion and WKB Method one can obation a good approximation for singularly perturbed problems.

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